Deriving the quartic formula

Consider the quartic equation in the form $a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 = 0$, where $a_4 \neq 0$.

1. Divide both sides of the equation by a_4:

$$\frac{a_0}{a_4} + \frac{a_1}{a_4} z + \frac{a_2}{a_4} z^2 + \frac{a_3}{a_4} z^3 + z^4 = 0$$

2. Substitute $z \leftarrow x - \frac{a_3}{4a_4}$ and expand:

$$\begin{aligned}
\frac{a_0}{a_4} + \frac{a_1}{a_4}\left(x - \frac{a_3}{4a_4}\right) + \frac{a_2}{a_4}\left(x - \frac{a_3}{4a_4}\right)^2 + \frac{a_3}{a_4}\left(x - \frac{a_3}{4a_4}\right)^3 + \left(x - \frac{a_3}{4a_4}\right)^4 &= 0 \\
\tfrac{a_0}{a_4} + \left(\tfrac{a_1 x}{a_4} - \tfrac{a_1 a_3}{4a_4^2}\right) + \left(\tfrac{a_2 x^2}{a_4} - \tfrac{a_2 a_3 x}{2a_4^2} + \tfrac{a_2 a_3^2}{16a_4^3}\right) + \left(\tfrac{a_3 x^3}{a_4} - \tfrac{3a_3^2 x^2}{4a_4^2} + \tfrac{3a_3^3 x}{16a_4^3} - \tfrac{a_3^4}{64a_4^4}\right) + \left(x^4 - \tfrac{a_3 x^3}{a_4} + \tfrac{3a_3^2 x^2}{8a_4^2} - \tfrac{a_3^3 x}{16a_4^3} + \tfrac{a_3^4}{256a_4^4}\right) &= 0 \\
\left(\frac{a_0}{a_4} - \frac{a_1 a_3}{4a_4^2} + \frac{a_2 a_3^2}{16a_4^3} - \frac{3a_3^4}{256a_4^4}\right) + \left(\frac{a_1}{a_4} - \frac{a_2 a_3}{2a_4^2} + \frac{a_3^3}{8a_4^3}\right)x + \left(\frac{a_2}{a_4} - \frac{3a_3^2}{8a_4^2}\right)x^2 + x^4 &= 0
\end{aligned}$$

3. For simplicity, define:

$$\begin{aligned}
b_0 &\equiv \frac{a_0}{a_4} - \frac{a_1 a_3}{4a_4^2} + \frac{a_2 a_3^2}{16a_4^3} - \frac{3a_3^4}{256a_4^4}, \\
b_1 &\equiv \frac{a_1}{a_4} - \frac{a_2 a_3}{2a_4^2} + \frac{a_3^3}{8a_4^3}, \\
b_2 &\equiv \frac{a_2}{a_4} - \frac{3a_3^2}{8a_4^2},
\end{aligned}$$

   reducing the equation to $b_0 + b_1 x + b_2 x^2 + x^4 = 0$.

4. Introduce an auxiliary variable w and add $\left(\frac{w+b_2}{2}\right)^2 - b_0 - b_1 x + w x^2$ to both sides to get a square term:

$$\begin{aligned}
\left(\frac{w+b_2}{2}\right)^2 + wx^2 + b_2 x^2 + x^4 &= \left(\frac{w+b_2}{2}\right)^2 - b_0 - b_1 x + wx^2 \\
\left(\frac{w+b_2}{2} + x^2\right)^2 &= \left(\frac{w+b_2}{2}\right)^2 - b_0 - b_1 x + wx^2
\end{aligned}$$

5. Add the zero term $\frac{b_1^2}{4w} - \frac{b_1^2}{4w}$ to the right-hand side to complete the square:

$$\begin{aligned}
\left(\frac{w+b_2}{2} + x^2\right)^2 &= \left(\frac{w+b_2}{2}\right)^2 - b_0 - \frac{b_1^2}{4w} + \frac{b_1^2}{4w} - b_1 x + wx^2 \\
\left(\frac{w+b_2}{2} + x^2\right)^2 &= \left(\frac{w+b_2}{2}\right)^2 - b_0 - \frac{b_1^2}{4w} + \left(-\frac{b_1}{2\sqrt{w}} + \sqrt{w}x\right)^2
\end{aligned}$$

6. Choose w such that $\left(\frac{w+b_2}{2}\right)^2 - b_0 - \frac{b_1^2}{4w} = 0$. This condition reduces to a cubic equation in w:

$$-\frac{b_1^2}{4} + \left(\frac{b_2^2}{4} - b_0\right)w + \frac{b_2}{2}w^2 + \frac{1}{4}w^3 = 0$$

   A solution using the cubic formula is given here. If $-\frac{p_0}{2} - \sqrt{\left(\frac{p_0}{2}\right)^2 + \left(\frac{p_1}{3}\right)^3} \neq 0$, where $p_0 = \frac{8b_0 b_2}{3} - b_1^2 - \frac{2b_2^3}{27}$ and $p_1 = -4b_0 - \frac{b_2^2}{3}$, then

$$w = \sqrt[3]{-\frac{p_0}{2} - \sqrt{\left(\frac{p_0}{2}\right)^2 + \left(\frac{p_1}{3}\right)^3}} - \frac{p_1}{3\sqrt[3]{-\frac{p_0}{2} - \sqrt{\left(\frac{p_0}{2}\right)^2 + \left(\frac{p_1}{3}\right)^3}}} - \frac{2b_2}{3}.$$

7. Factor the equation as a difference of squares:

$$\left(\frac{w+b_2}{2}+x^2\right)^2-\left(-\frac{b_1}{2\sqrt{w}}+\sqrt{w}x\right)^2=0$$
$$\left(\frac{b_1}{2\sqrt{w}}+\frac{w+b_2}{2}-\sqrt{w}x+x^2\right)\left(-\frac{b_1}{2\sqrt{w}}+\frac{w+b_2}{2}+\sqrt{w}x+x^2\right)=0$$

Solving the resulting quadratic equations for x, combining the solutions, and back-substituting yields all four solutions to the quartic equation:

$$z=\frac{m\sqrt{w}+n\sqrt{-2b_2-w-m\times\frac{2b_1}{\sqrt{w}}}}{2}-\frac{a_3}{4a_4},\text{ where } m,n\in\{-1,1\}.$$

This formula fails if $w=0$, which implies $b_1=0$. In this case, the quartic equation reduces to a quadratic equation in x^2, and the solutions are:

$$z=m\sqrt{\frac{-b_2+n\sqrt{b_2^2-4b_0}}{2}}-\frac{a_3}{4a_4},\text{ where } m,n\in\{-1,1\}.$$

Alternative solution for semi-palindromic quartic equations
Consider the semi-palindromic quartic equation $a_4s^2+a_3sz+a_2z^2+a_3z^3+a_4z^4=0$, where $a_4\neq0$.

1. Divide through by z^2:
$$\frac{a_4s^2}{z^2}+\frac{a_3s}{z}+a_2+a_3z+a_4z^2=0$$

2. Add $2a_4s$ to both sides of the equation:
$$\frac{a_4s^2}{z^2}+\frac{a_3s}{z}+a_2+2a_4s+a_3z+a_4z^2=2a_4s$$

3. Rearrange the left-hand side as a quadratic in $\frac{s}{z}+z$:
$$a_2+\frac{a_3s}{z}+a_3z+\frac{a_4s^2}{z^2}+2a_4s+a_4z^2=2a_4s$$
$$a_2+a_3\left(\frac{s}{z}+z\right)+a_4\left(\frac{s}{z}+z\right)^2=2a_4s$$

4. Subtract $2a_4s$ from both sides to isolate the quadratic expression:
$$a_2-2a_4s+a_3\left(\frac{s}{z}+z\right)+a_4\left(\frac{s}{z}+z\right)^2=0$$

5. Solve for $\frac{s}{z}+z$ using the quadratic formula:
$$\frac{s}{z}+z=\frac{-a_3-\sqrt{a_3^2-4a_2a_4+8a_4^2s}}{2a_4}\text{ and }\frac{s}{z}+z=\frac{-a_3+\sqrt{a_3^2-4a_2a_4+8a_4^2s}}{2a_4}.$$

For each solution, the equation can be rearranged into a quadratic in z. Solving each equation for z using the quadratic formula and combining the solutions gives the four roots of the semi-palindromic quartic equation:

$$z=\frac{-a_3+m\sqrt{a_3^2-4a_2a_4+8a_4^2s}+n\sqrt{\left(a_3-m\sqrt{a_3^2-4a_2a_4+8a_4^2s}\right)^2-16a_4^2s}}{4a_4},\text{ where } m,n\in\{-1,1\}.$$